UDC 539.3

ON THE CLOSED SOLUTION OF THE FIRST BOUNDARY VALUE PROBLEM OF ELASTICITY THEORY FOR A SPACE WITH A SPHERICAL CAVITY

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A solution of the problem concerning the strain of an isotropic elastic space with a spherical cavity by arbitrary normal loads applied to the cavity surface is obtained in quadratures by the method elucidated in [1, 2]. The corresponding axisymmetric problem is solved for arbitrary normal loads and tangential loads acting in the meridian plane. As an illustration, a solution is presented of the problem for an axisymmetric load distributed uniformly on an infinitely thin ring, and for the case of a concentrated normal force.

1. Let us perform the investigation in a spherical r, θ, φ coordinate system. Let us first analyze the axisymmetric problem of the deformation of an isotropic elastic space $r \geqslant R$ with a spherical cavity by arbitrary normal $\sigma(\theta)$ and tangential $\tau(\theta)$ loads applied to the sphere $r = R$.

It is known [3] that the problem of the equilibrium of an axisymmetrically loaded body of revolution decomposes into two self-consistent problems: the problem of torsion relative to the axis of symmetry and the problem of deformation in the meridian plane. Let us limit ourselves to an analysis of the latter problem.

Let us represent the external loads given on the sphere $r = R$ as series [3] ($P_n(x)$ are Legendre polynomials)

$$\sigma(\theta) = \sum_{n=0}^{\infty} \sigma_n P_n(\cos\theta), \quad \tau(\theta) = \sum_{n=1}^{\infty} \tau_n \frac{dP_n(\cos\theta)}{d\theta} \tag{1.1}$$

$$\sigma_n = \frac{2n+1}{2} \int_0^{\pi} \sigma(\theta) P_n(\cos\theta) \sin\theta \, d\theta \tag{1.2}$$

$$\tau_n = \frac{2n+1}{2n(n+1)} \int_0^{\pi} \tau(\theta) \frac{dP_n(\cos\theta)}{d\theta} \sin\theta \, d\theta$$

The boundary conditions on the sphere $r = R$ are

$$\sigma_r = -\sigma(\theta), \quad \tau_{r\theta} = -\tau(\theta), \quad \tau_{r\varphi} = 0 \tag{1.3}$$

The solution of the equilibrium equations in displacements which vanishes at infinity can be represented as [3]

$$u_r = \sum_{n=0}^{\infty} \left[\frac{A_n}{r^n} n(n+3-4\nu) - \frac{B_n}{r^{n+1}}(n+1) \right] P_n(\cos\theta) \tag{1.4}$$

$$u_\theta = \sum_{n=1}^{\infty} \left[\frac{A_n}{r^n}(-n+4-4\nu) + \frac{B_n}{r^{n+2}} \right] \frac{dP_n(\cos\theta)}{d\theta}$$

Substituting the stresses σ_r and $\tau_{r\theta}$ determined by means of the displacements u_r and

u_θ into the boundary conditions and taking (1.1) into account, we obtain a system of two linear algebraic equations for the constants A_n and B_n ($n = 1, 2, \ldots$)

$$\frac{\sigma_n}{2G} = A_n \frac{n(n^2+3n-2\nu)}{R^{n+1}} - B_n \frac{(n+1)(n+2)}{R^{n+3}} \tag{1.5}$$

$$\frac{\tau_n}{2G} = A_n \frac{n^2-2+2\nu}{R^{n+1}} + B_n \frac{n+2}{R^{n+3}}$$

Here G is the shear modulus, and ν is the Poisson's ratio. The determinant of this system is not zero for all values of $n = 1, 2, \ldots$

The solution of the system (1.5) is the following

$$A_n = \frac{\sigma_n + (n+1)\tau_n}{4G\Delta} R^{n+1}, \qquad \Delta = n^2 + (1-2\nu)n + 1 - \nu \tag{1.6}$$

$$B_n = \frac{\sigma_n(n^2-2+2\nu) + \tau_n n(n^2+3n-2\nu)}{4G\Delta(n+2)} R^{n+3}$$

Substituting the values found for the coefficients A_n and B_n into (1.4) and taking account of the relationship (1.2), we obtain

$$u_r = I_{12}, \quad u_\theta = I_{34} \tag{1.7}$$

$$I_{sk} = \frac{R}{2\pi G}\int_0^\pi [\sigma(\alpha) H_s + \tau(\alpha) H_k] \sin\alpha \, d\alpha$$

Here

$$\frac{4}{\pi} H_1 = x^2 + S_{12}^{(A)} P_n(\cos\theta) P_n(\cos\alpha) \tag{1.8}$$

$$\frac{4}{\pi} H_2 = S_{12}^{(B)} P_n(\cos\theta) \frac{dP_n(\cos\alpha)}{d\alpha}$$

$$\frac{4}{\pi} H_3 = S_{34}^{(A)} \frac{dP_n(\cos\theta)}{d\theta} P_n(\cos\alpha)$$

$$\frac{4}{\pi} H_4 = S_{34}^{(B)} \frac{dP_n(\cos\theta)}{d\theta} \frac{dP_n(\cos\alpha)}{d\alpha}$$

$$S_{jk}^{(A)} = \sum_{n=1}^{\infty} (A_{jk} x^n + A_{kn} x^{n+2}), \qquad x = \frac{R}{r}$$

$$S_{jk}^{(B)} = \sum_{n=1}^{\infty} (B_{jk} x^n + B_{kn} x^{n+2})$$

$$A_{1n} = \frac{n(2n+1)(n+3-4\nu)}{\Delta}, \qquad A_{2n} = -\frac{(2n+1)(n+1)(n^2-2+2\nu)}{(n+2)\Delta} \tag{1.9}$$

$$B_{1n} = \frac{(2n+1)(n+3-4\nu)}{\Delta}, \qquad B_{2n} = -\frac{(2n+1)(n^2+3n-2\nu)}{(n+2)\Delta}$$

$$A_{3n} = -\frac{(2n+1)(n-4+4\nu)}{\Delta}, \qquad A_{4n} = \frac{(2n+1)(n^2-2+2\nu)}{(n+2)\Delta}$$

$$B_{3n} = -\frac{(2n+1)(n-4+4\nu)}{n\Delta}, \qquad B_{4n} = \frac{(2n+1)(n^2+3n-2\nu)}{(n+1)(n+2)\Delta}$$

The coefficients A_{in} and B_{in} ($i = 1, 2, 3, 4$) can be represented as follows (a and $\bar{a}$ are complex-conjugate roots of the equation $\Delta = 0$):

$$A_{1n} = (2n+1) + 4(1-\nu) + \frac{C_1}{n-a} + \frac{\bar{C}_1}{n-\bar{a}} \tag{1.10}$$

$$A_{2n} = -(2n+1) + 4(1-\nu) - \frac{2}{n+2} + \frac{C_2}{n-a} + \frac{\bar{C}_2}{n-\bar{a}}$$

$$B_{1n} = 2 + \frac{D_1}{n-a} + \frac{\bar{D}_1}{n-\bar{a}}, \quad B_{2n} = -2 - \frac{2}{n+2} + \frac{D_2}{n-a} + \frac{\bar{D}_2}{n-\bar{a}}$$

$$A_{3n} = -2 + \frac{C_3}{n-a} + \frac{\bar{C}_3}{n-\bar{a}}, \quad A_{4n} = 2 - \frac{2}{n+2} + \frac{C_4}{n-a} + \frac{C_4}{n-\bar{a}}$$

$$B_{3n} = \frac{4}{n} + \frac{D_3}{n-a} + \frac{\bar{D}_3}{n-\bar{a}}$$

$$B_{4n} = \frac{2}{n+1} - \frac{2}{n+2} + \frac{D_4}{n-a} + \frac{\bar{D}_4}{n-\bar{a}}$$

$$a = -\frac{1-2\nu}{2} + i\,\frac{\sqrt{3-4\nu^2}}{2}$$

The quantities C_i and D_i $(i = 1, 2, 3, 4)$ depend only on the Poisson's ratio

$$C_1 = -2 + 6\nu - 4\nu^2 + i\,\frac{3 + \nu - 12\nu^2 + 8\nu^3}{\sqrt{3-4\nu^2}}$$

$$C_2 = 1 + 2\nu - 4\nu^2 + i\,\frac{2 - 5\nu - 4\nu^2 + 8\nu^3}{\sqrt{3-4\nu^2}}$$

$$D_1 = \frac{1}{2}\left(5 - 4\nu + i\,\frac{3 - 10\nu + 8\nu^2}{\sqrt{3-4\nu^2}}\right)$$

$$D_2 = C_2 - C_1 - D_1, \quad C_3 = -3C_4, \quad C_4 = \frac{C_2 - C_1}{2}$$

$$D_3 = -3D_4, \quad D_4 = 1 + i\,\frac{2(1-\nu)}{\sqrt{3-4\nu^2}}$$

Let us convert (1.8) for the functions H_i $(i = 1, 2, 3, 4)$ by using the relationship (which follows from the theorem for addition of spherical functions)

$$P_n(\cos\theta)\,P_n(\cos\alpha) = \frac{2}{\pi}\int_0^{\pi/2} P_n(\lambda)\,d\psi$$

We obtain

$$\lambda = \cos(\theta + \alpha) + 2\sin\theta\sin\alpha\sin^2\psi$$

$$H_1 = \frac{\pi}{4}x^2 + \frac{1}{2}\int_0^{\pi/2} S_{12}^{(A)} P_n(\lambda)\,d\psi \quad H_2 = \frac{1}{2}\frac{\partial}{\partial\alpha}\int_0^{\pi/2} S_{12}^{(B)} P_n(\lambda)\,d\psi \tag{1.11}$$

$$H_3 = \frac{1}{2}\frac{\partial}{\partial\theta}\int_0^{\pi/2} S_{34}^{(A)} P_n(\lambda)\,d\psi \quad H_4 = \frac{1}{2}\frac{\partial^2}{\partial\theta\,\partial\alpha}\int_0^{\pi/2} S_{34}^{(B)} P_n(\lambda)\,d\psi$$

The series in (1.11) can be summed if (1.10) is taken into account and the following formulas are used [1, 4]:

$$\sum_{n=0}^{\infty} x^n P_n(\lambda) = \frac{1}{s(x,\lambda)}, \quad s(x,\lambda) = \sqrt{x^2 - 2x\lambda + 1}$$

$$\sum_{n=1}^{\infty} n x^n P_n(\lambda) = x\frac{\partial}{\partial x}\frac{1}{s(x,\lambda)}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} P_n(\lambda) = \frac{1}{x}\int_0^x dx \sum_{n=1}^{\infty} x^n P_n(\lambda) = \int_0^1 \frac{1}{y}\left[\frac{1}{s(xy,\lambda)} - 1\right] dy$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n-a} P_n(\lambda) = x^a \int_0^x dx \sum_{n=0}^{\infty} x^{n-a-1} P_n(\lambda) = \int_0^1 \frac{y^{-(1+a)}}{s(xy, \lambda)} dy, \qquad \operatorname{Re} a < 0$$

To the accuracy of a factor $-1/a$ the last integral agrees with a hypergeometric function of two variables $F_1(-a, 1/2, 1/2, 1-a; xe^{i \arccos \lambda}, xe^{-i \arccos \lambda})$, as follows from its integral representation [5].

Also taking into account that [4]

$$\int_0^{\pi/2} \frac{d\psi}{s(x, \lambda)} = \frac{K(k)}{h}, \qquad h^2 = x^2 - 2x\cos(\theta + \alpha) + 1$$

$$k^2 h^2 = 4x \sin\theta \sin\alpha$$

where $K(k)$ is the complete elliptic integral of the first kind, we finally obtain

$$H_1 = -\frac{5-4\nu}{4}\pi - \frac{1-2\nu}{2}\pi x^2 + 2(1-\nu)(1+x^2)\frac{K(k)}{h} + \tag{1.12}$$

$$\frac{1-x^2}{2}\left[\frac{K(k)}{h} + 2x\frac{\partial}{\partial x}\frac{K(k)}{h}\right] + \operatorname{Re}\int_0^1 \left(\frac{C_1 + C_2x^2}{y^{1+a}} - x^2y\right)\left[\frac{K(k_1)}{h_1} - \frac{\pi}{2}\right] dy$$

$$H_2 = (1-x^2)\frac{\partial}{\partial\alpha}\frac{K(k)}{h} + \operatorname{Re}\int_0^1 \left(\frac{D_1 + D_2x^2}{y^{1+a}} - x^2y\right)\frac{\partial}{\partial\alpha}\frac{K(k_1)}{h_1} dy$$

$$H_3 = (x^2-1)\frac{\partial}{\partial\theta}\frac{K(k)}{h} + \operatorname{Re}\int_0^1 \left(\frac{C_3 + C_4x^2}{y^{1+a}} - x^2y\right)\frac{\partial}{\partial\theta}\frac{K(k_1)}{h_1} dy$$

$$H_4 = \operatorname{Re}\int_0^1 \left[\frac{D_3 + D_4x^2}{y^{1+a}} + \frac{2}{y} + x^2(1-y)\right]\frac{\partial^2}{\partial\theta\,\partial\alpha}\frac{K(k_1)}{h_1} dy$$

$$h_1^2 = x^2y^2 - 2xy\cos(\theta+\alpha) + 1, \quad k^2h^2 = 4xy\sin\theta\sin\alpha$$

Therefore, the solution (1.7) of the first axisymmetric boundary value problem of elasticity theory for a space with a spherical cavity is represented in quadratures.

Using the properties of the complete elliptic integrals of the first and second kinds and of their derivatives [4], it can be shown that the functions $H_i(x, \theta, \alpha)$ $(i = 1, 2, 3, 4)$ are continuous for $r > R$ $(0 < x < 1)$. They have discontinuities at $\alpha = \theta$ on the sphere $r = R$.

As $\alpha \to \theta$, the following representations hold:

$$H_{1,4}(1, \theta, \alpha)\sin\alpha = -2(1-\nu)\ln|\theta - \alpha| + Q_{1,4}(\theta, \alpha)$$
$$H_{2,3}(1, \theta, \alpha)\, 2\sin\alpha = (1-2\nu)\pi \operatorname{sgn}(\alpha - \theta) + Q_{2,3}(\theta, \alpha)$$

The functions $Q_i(\theta, \alpha)$ $(i = 1, 2, 3, 4)$ are bounded and contonuous for $0 < \theta, \theta < \pi$.

Let us note the following properties of the functions H_i:

$$H_{1,4}(x, \theta, \pi - \alpha) = H_{1,4}(x, \pi - \theta, \alpha), \quad H_{2,3}(x, \theta, \pi - \alpha) = -H_{2,3}(x, \pi - \theta, \alpha)$$

2. As an example, let us consider the problem of the deformation of an elastic space $r \geqslant R$ with a spherical cavity by a normal force P and a tangential force Q distributed uniformly in an infinitely thin ring $r = R, \theta = \theta_0$. The load distributed along a line should certainly be considered as the limit value of the corresponding surface load.

The boundary conditions for $r = R$ are ($\delta(x)$ is the Dirac delta function)

$$\sigma_r = -\frac{P\delta(\alpha-\theta_0)}{2\pi R^2 \sin\theta_0}, \quad \tau_{r\theta} = -\frac{Q\delta(\alpha-\theta_0)}{2\pi R^2 \sin\theta_0}, \quad \tau_{r\varphi} = 0 \tag{2.1}$$

Substituting the values $\sigma(\alpha) = -\sigma_r$ and $\tau(\alpha) = -\tau_{r\theta}$ given by the relationships (2.1) into the solution (1.7), we obtain

$$u_r = I_{12}^\circ, \quad u_\theta = I_{34}^\circ$$

$$I_{jk}^\circ = \frac{1}{4\pi^2 RG}[PH_j(x,\theta,\theta_0) + QH_k(x,\theta,\theta_0)]$$

The functions H_i ($i = 1, 2, 3, 4$) are given by (1.12) in which we should set $\alpha = \theta_0$. It is seen from (1.7) that the functions H_i are Green's functions of the first axisymmetric boundary value problem of elasticity theory for a space with a spherical cavity.

3. Let us consider the problem of the deformation of an elastic space $r \geqslant R$ with a spherical cavity by a normal concentrated force P applied at the pole $\theta = 0$ of the sphere $r = R$.

The boundary conditions at $r = R$ are

$$\sigma_r = -\frac{P\delta(\theta)}{\pi R^2 \sin\theta}, \quad \tau_{r\theta} = \tau_{r\varphi} = 0$$

According to (1.7), the solution of the problem is

$$u_r = \frac{P}{2\pi^2 RG} H_1(x,\theta,0), \quad u_\theta = \frac{P}{2\pi^2 RG} H_3(x,\theta,0)$$

Here, in conformity with (1.12)

$$\frac{2}{\pi} H_1(x, \theta, 0) = 1 + x^2 - S + \frac{1-S}{2S}[1 - x^2 + 4(1+x^2)(1-\nu)] -$$
$$\frac{x(1-x^2)(x-\cos\theta)}{S^3} - \cos\theta \ln\frac{S + x - \cos\theta}{1-\cos\theta} +$$
$$\operatorname{Re}\frac{C_1 + C_2 x^2}{a}\left[1 - F_1\left(-a, \frac{1}{2}, \frac{1}{2}, 1-a; xe^{i\theta}, xe^{-i\theta}\right)\right]$$

$$\frac{2}{\pi\sin\theta} H_3(x, \theta, 0) = \frac{x(1-x^2)}{S^3} + \frac{(1-2\sin^2\theta)x - (1-S)\cos\theta}{S\sin^2\theta} +$$
$$\ln\frac{S + x - \cos\theta}{1-\cos\theta} - x\operatorname{Re}\frac{C_3 + C_4 x^2}{1-a} F_1\left(1-a, \frac{3}{2}, \frac{3}{2}, 2-a, xe^{i\theta}, xe^{-i\theta}\right)$$
$$S = \sqrt{x^2 - 2x\cos\theta + 1}$$

4. The solution obtained for the problem of the effect of a concentrated force is a Green's function of the non-axisymmetric boundary value problem for an elastic space with a spherical cavity.

If an arbitrary normal load $\sigma = \sigma(\theta, \varphi)$ is applied to the surface $r = R$, then the solution is $u_r = J_1, \quad u_\theta = J_3$

$$J_k = \frac{R}{4\pi^2 G}\int_0^{2\pi} d\beta \int_0^{\pi} \sigma(\alpha, \beta) H_k\left(\frac{R}{r}, \theta^\circ, 0\right) \sin\alpha \, d\alpha$$

$$\theta^\circ = \arccos[\cos\theta\cos\alpha + \sin\theta\sin\alpha\cos(\varphi - \beta)]$$

Therefore, a closed solution has been obtained for the problem of deformation of an elastic space with a spherical cavity by an arbitrary non-axisymmetric normal load applied to the cavity surface.

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ON THE EQUATIONS OF MAGNETOELASTIC THIN PLATES

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Hypotheses relative to the character of variation of the electromagnetic field and elastic displacements over the thickness of a plate were formulated in [1, 2] on the basis of solutions obtained by the method of asymptotic integration of the three-dimensional equations of magnetoelasticity. Two-dimensional equations of magnetoelasticity, in which unknown boundary values of the components induced by the electromagnetic field enter, have been obtained on the basis of these hypotheses. The equations obtained must hence be examined in combination with the Maxwell equations for the medium surrounding the plate under general boundary conditions at the interface of the two media. This means that the magnetoelasticity problem nevertheless remains three-dimensional.

On the basis of the mentioned hypotheses for the magnetoelasticity of thin bodies [1, 2], an attempt is made in this paper to reduce the three-dimensional magnetoelasticity problem to a two-dimensional problem, which will substantially facilitate the investigation of questions about the magnetoelasticity of thin bodies.

1. Let an isotropic plate of constant thickness $2h$, fabricated from a material with finite electrical conductivity, be in an external stationary magnetic field with a given magnetic induction vector $\mathbf{B}_0 = (B_{0x}, B_{0y}, B_{0z})$. The problem is solved under the assumption that the Maxwell equations for a vacuum are valid for the medium surrounding the plate. It is also assumed that the influence of displacement currents on the elastic vibrations characteristics can be neglected.

The elastic and electromagnetic properties of the plate material are characterized